POLYNOMIAL IDENTITIES OF RELATED RINGS

BY

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ABSTRACT

The set of polynomial identities of a ring A is considered, as well as some types of minimal identities. The change which occurs in these identities upon passage to related rings is then studied.

In this paper we are concerned with the set of all polynomial identities of a ring. We call two rings equivalent if their sets of identities (over some fixed domain of operators) coincide. It is first shown that the tensor product preserves this equivalence, and in particular the matrix rings over equivalent rings are equivalent. We then consider two types of "minimal" identities (not necessarily of minimal degree) and show that in some respects all rings with identities behave like matrix rings. As a corollary, we get the converse of a recent theorem by Procesi and Small [5]. Finally, we go down from A_n to A_k with $k < n$, A an arbitrary ring, and present a method of reducing identities in this process.

1. The set of identities of related rings. Let Ω be a fixed domain of operators which we assume to be a commutative ring with unity. Polynomial identities as well as tensor products and homomorphisms are understood throughout to be taken over Ω. Every ring considered is assumed to be an Ω-algebra and to satisfy a polynomial identity, and we assume, moreover, that at least one of the coefficients of this identity is equal to 1. This assumption persists in homomorphic images, and is not very restrictive as most of the identities one encounters in applications have only 1 and -1 for their coefficients. Also it is known ([2]) that if Ω is integral domain then any PI-ring A, for which $\alpha a = 0$ with $\alpha \in \Omega$, $a \in A$ implies $\alpha = 0$ or $a = 0$, satisfies such an identity (namely, a power of the standard

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identity).* We denote the set of identities of a ring A by $I(A)$, and we say that two rings A and B are *equivalent* (over Ω) if $I(A) = I(B)$. We write this as $A \equiv B$. Epi- and monomorphisms will be written as $\twoheadrightarrow$ and $\rightarrowtail$ respectively.

The following is the main theorem of this section.

THEOREM 1. *Assume that Ω is a field. If $A \equiv A'$ and $B \equiv B'$ then $A \otimes B \equiv A' \otimes B'$.*

Note that this theorem shows that the relation $\equiv$ is actually a congruence relative to the tensor product, and therefore the equivalence classes of Ω-algebras form a commutative semigroup with unity.

For the proof we use the universal rings introduced by Amitsur. See [2] for details. Using Amitsur's technique we first give a simple criterion for equivalence, which does not use the notion of identities. For a ring A and a set I we write A^I for the I-direct power of A, i.e. the direct product of A with itself I times.

LEMMA 2. *Let A be any ring, U the universal ring of A. Then for some set I we have the diagram*

$$A^I \leftarrowtail U \twoheadrightarrow A.$$

Conversely, if for some rings A and B and a set I we have a diagram

$$A^I \leftarrowtail B \twoheadrightarrow A,$$

then $A \equiv B$.

PROOF. Let $\mathscr{F}$ be the free ring over the given domain of operators Ω, generated by a sufficiently large set of noncommutative indeterminates, so that $U = \mathscr{F}/I(A)$. If $\phi : \mathscr{F} \twoheadrightarrow A$ is any epimorphism, then ker $\phi \supseteq I(A)$ and therefore there is a unique map $\bar{\phi} : U \twoheadrightarrow A$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathscr{F} & \longrightarrow & U \\ & {\scriptstyle\phi} \searrow \quad \swarrow {\scriptstyle\bar{\phi}} & \\ & A & \end{array}$$

Let I be the set of all epimorphisms $\phi : \mathscr{F} \twoheadrightarrow A$. For $u \in U$ we define $i(u)$ to be the function in A^I such that $i(u)(\phi) = \bar{\phi}(u)$ for all $\phi \in I$. Suppose $i(u) = 0$ for some $u \in U$ and let $u = \bar{f} = f + I(A)$ with $f \in \mathscr{F}$. Then for every $\phi \in I$ we have $\phi(f)$

* The restriction appearing in [2], i.e. $aA = 0$ implies $a = 0$, is actualy weaker, but this does not seem to suffice for the proof of Lemma 2 there and its consequences.

$= \bar{\phi}(u) = 0$ so that $f \in \bigcap_{\phi \in I} \ker \phi = I(A)$, whence $u = \bar{f} = 0$. Thus i is a monomorphism and so, all the other parts of the Lemma being trivial, the proof is completed.

Proof of Theorem 1. Let U be the common universal ring of A and A', so that in particular $A \equiv A' \equiv U$, and consider the diagram

$$A^I \leftarrowtail U \twoheadrightarrow A.$$

This diagram, upon tensoring by B, induces the diagram

$$A^I \otimes B \leftarrowtail U \otimes B \twoheadrightarrow A \otimes B,$$

and since $A^I \otimes B$ is naturaly embedded in $(A \otimes B)^I$, also the diagram

$$(A \otimes B)^I \leftarrowtail U \otimes B \twoheadrightarrow A \otimes B,$$

whence $A \otimes B \equiv U \otimes B$. Of course we also have $A' \otimes B \equiv U \otimes B$, and so $A \otimes B \equiv A' \otimes B$. In the same way $A' \otimes B \equiv A' \otimes B'$ and we conclude that $A \otimes B \equiv A' \otimes B'$.

If Ω is not assumed to be a field, we can still pass from the diagram $A^I \leftarrowtail U \twoheadrightarrow A$ in the proof of Theorem 1 to the diagram

$$(A_k)^I \cong (A^I)_k \leftarrowtail U_k \twoheadrightarrow A_k,$$

where A_k denotes the matrix ring of order k over A. Hence we have

THEOREM 3. *For arbitrary rings A and B, $A \equiv B$ implies $A_k \equiv B_k$.*

There is an alternative, straightforward method to prove the results of this section, which has some independent interest. We demonstrate this method by reproving Theorem 3. Suppose $A \equiv B$ over Ω, and let $f(x_1, \cdots, x_r)$ be an identity of A_k. Let $\Omega[x]$ be the free ring generated over Ω by an infinite number of non-commutative indeterminates, and consider in $\Omega[x]_k$ the matrices of indeterminates $X^{(\nu)} = (x_{ij}^{(\nu)})$, $\nu = 1, \cdots, r$. We substitute these matrices in f and write

$$f(X^{(1)}, \cdots, X^{(r)}) = \begin{bmatrix} f_{11}(x) & \cdots & f_{1k}(x) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ f_{k1}(x) & \cdots & f_{kk}(x) \end{bmatrix} \in \Omega[x]_k,$$

where the $f_{ij}(x)$ are polynomials over Ω in the rk^2 indeterminates $x_{ij}^{(\nu)}$. If we specialize in this relation $x_{ij}^{(\nu)} = a_{ij}^{(\nu)} \in A$, then since f is an identity for A_k, the lefthand side vanishes. Therefore the righthand side also vanishes and we may conclude that the $f_{ij}(x)$ are identities of A. Since $A \equiv B$, $f_{ij}(x)$ are identities of B as well. But then reversing the process, we see that every substitution $x_{ij}^{(\nu)} = b_{ij}^{(\nu)} \in B$ annihilates the righthand side and so also the lefthand side of the above relation. Since every substitution from B_k in f is of this form, it follows that f is an identity of B_k.

If Ω is a field, one can similarly prove that $A \equiv A'$ implies $A \otimes B \equiv A' \otimes B$ (from which Theorem 1 follows) by replacing $\Omega[x]_k$ in the present proof by $\Omega[x] \otimes B$.

We remark finaly that besides the equivalence relation $\equiv$, one can also define a preordering on rings by declaring $A \leqq B$ whenever $I(A) \supseteq I(B)$. All the results of this section, and some of the next, generalize easily to this case. In particular, if Ω is a field, this relation is preserved by tensor products and thus it induces a partial ordering on the semigroup of equivalence classes of algebras modulo $\equiv$. It is clear also that the relation $\leqq$ is preserved by μ^* (see below).

2. Minimal identities of related rings. In what follows we assume for convenience that all our rings posses a unit element, though this is not necessary. Let $s_k(x_1, \cdots, x_k)$ be the standard identity of degree k, and note that if A satisfies s_k with k minimal, then k must be even. For if k were odd, then $s_k(x_1, \cdots, x_{k-1}, 1) = s_{k-1}(x_1, \cdots, x_{k-1})$ and thus k would not be minimal. The same remark holds for powers of standard identities. We write $\mu(A) = n$ if A satisfies s_{2n} and no s_k with $k < 2n$. If A satisfies no standard identity we write $\mu(A) = \infty$. We write in addition $\mu^*(A) = n$ if A satisfies a power of s_{2n}, and no power of s_k with $k < 2n$. As we shall see later (Corollary 14), our assumption on the identity satisfied by A implies that A satisfies a power of some standard identity, so that $\mu^*(A)$ is always finite. Finally, we call a ring A *M_n-ring* if $A \equiv C_n$ over Ω for some commutative Ω-algebra C, and *M-ring* if A is an M-ring for some n.

The theorem of Amitsur and Levitzki [1] states that $\mu(C_n) = n$ for a commutative ring C, and from this it clearly follows that

$$(*) \quad \text{if } A = C_n \text{ then } \mu(A_k) = k\mu(A).$$

Our aim in this section is to generalize (*) in several directions. First we note that (*) can be taken to mean that for $A = C_n$, $\mu(A \otimes \Omega_k) = \mu(A) \cdot \mu(\Omega_k)$. We generalize this in the following

THEOREM 4. *Assume that Ω is a field and let A and B be M_n- and M_k-algebras. Then $A \otimes B$ is M_{nk}-algebra and $\mu(A \otimes B) = \mu(A) \cdot \mu(B)$.*

PROOF. Since $A \equiv C_n$ and $B \equiv C'_k$, we have $A \otimes B \equiv C_n \otimes C'_k \cong (C \otimes C')_{nk}$, so that $A \otimes B$ is an M_{nk}-algebra. The rest of the theorem is now trivial.

COROLLARY 5. *Let Ω be a field and let S be the semigroup of equivalence classes of algebras modulo the relation $\equiv$. Then $T = \{[A] \in S \mid A$ is an M-algebra$\}$ is a subsemigroup and μ induces an homomorphism of T onto the multiplicative semigroup of the natural numbers.*

An analogous generalization of (*) follows from Theorem 3.

COROLLARY 6. *If A is an M_n-ring then A_k is an M_{nk}-ring and $\mu(A_k) = k\mu(A)$.*

The class of M-rings contains many rings besides matrix rings over commutative rings, for we have the following

LEMMA 7. *Any prime ring A is an M-ring.*

PROOF. The proof follows immediately from [3]. Recall that A satisfies an identity having 1 for one of its coefficients, and this identity is nontrivial in the sense of [3].

If $A = [GF(q)]_k$ is a matrix ring over a finite field then, trivially, A is an M_k-ring taking $C = GF(q)$. Suppose now that A is not of the form $[GF(q)]_k$ and let Q be its ring of quotients. Then it is shown in [3] that for every commutative ring K, $A \equiv Q \equiv Q \otimes K$ over Ω, where the tensor product is arbitrary. If in particular we choose for K a splitting field of Q, and form the tensor product over the center of Q, we obtain $A \equiv K_n$, i.e. A is an M-ring. (One checks easily that in both cases C and K are indeed Ω-algebras.)

We remark that if one puts the further restriction that Ω be an infinite integral domain and that $\alpha a = 0$ with $\alpha \in \Omega$ $a \in A$ implies $\alpha = 0$ or $a = 0$, then Lemma 7 can be extended to semiprime rings. This follows from Theorems 2 and 6 of [2].

The converse of Lemma 7 is false, as demonstrated by the following example of an M-ring which has nilpotent ideals. Let F be an infinite field and let $A = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} \middle| \alpha, \beta \in F \right\}$. Then A is a commutative ring with unity embedding F, and $N = \left\{ \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix} \middle| \beta \in F \right\}$ is a nilpotent ideal of A. By Lemma 6 of [2] $A \equiv F$ and by our Theorem 3 $A_k \equiv F_k$ for every $k \geqq 1$.

Thus A_k is an M_k-algebra with unity, having N_k as a nilpotent ideal. Moreover, Amitsur has given an example of an M-ring which is not embeddable in C_n with C commutative [6].

We now take over μ^* as another generalization of (*). Our main result in this direction is

THEOREM 8. *For any ring A $\mu^*(A_k) = k\mu^*(A)$.*

REMARK. The inequality "$\leqq$" follows from the theorem of Procesi and Small [5].

To prove Theorem 8, we need two lemmas.

LEMMA 9. *If A is any M-ring then $\mu^*(A) = \mu(A)$.*

PROOF. It is clearly enough to prove the result for $A = C_n$, C commutative. Since we know that $\mu(C_n) = n$ and $\mu^*(C_n) \leqq \mu(C_n)$, we have only to show that if $k < n$ then C_n satisfies no power of $s_{2k}(x_1, \cdots, x_{2k})$. Indeed, let e_{ij} be n^2 matrix units and put

$$Q = s_{2k}(e_{12}, e_{23}, e_{34}, \cdots, e_{kk+1}, e_{k+1k}, \cdots, e_{43}, e_{32}, e_{21}).$$

We first take a look at any non-zero term of Q, which must have the form $e_{i_1i_2}e_{i_2i_3}e_{i_3i_4}\cdots e_{i_ri_{r+1}}$, and call i_1 and i_{r+1} "external" indices in this term and all the rest "internal". It is now seen that each internal index appears in this term an even number of times, whereas the external indices, unless they are equal, have an odd number of appearances. Next we set $i_1 = 1$ and observe that this yields a single non-zero term for Q, namely $e_{12}e_{23}\cdots e_{kk+1}e_{k+1k}\cdots e_{32}e_{21} = e_{11}$. By the above considerations (or by actual counting) we see that each index has here an even number of appearances, and therefore the same is true for *every* term of Q. Hence every other non-zero term must have its external indices equal, and thus has the value e_{ii} for some $i > 1$. It follows that the matrix Q is diagonal, and as it has 1 in its uppermost left corner, it is not nilpotent. This shows that no power of s_{2k} is an identity for C_n, hence the proof is completed.

The next lemma shows that a semiprime ring, while it may not be an M-ring, still inherits from its prime images two important properties of M-rings.

LEMMA 10. *If A has no non-zero nilpotent ideals then $\mu(A_k) = k\mu(A)$ and $\mu^*(A) = \mu(A)$.*

PROOF. First, if A is prime then by Lemma 7 A is an M-ring, and the two assertion follow from Corollary 6 and Lemma 9. If now A is any ring having no non-zero nilpotent ideals then A is a subdirect product of prime rings $A^{(i)}$ and thus also A_k is a subdirect product of $A_k^{(i)}$. Since A satisfies a nontrivial identity which remains nontrivial in each of the $A^{(i)}$, the above results are applicable.

It also follows that the $\mu(A^{(i)})$ are bounded. Letting r be an index such that $\mu(A^{(i)}) \leqq \mu(A^{(r)})$ for every i, we have $\mu(A) = \mu(A^{(r)})$. Since $\mu(A_k^{(i)}) = k\mu(A^{(i)})$ we also have $\mu(A_k^{(i)}) \leqq \mu(A_k^{(r)})$ and so $\mu(A_k) = \mu(A_k^{(r)}) = k\mu(A^{(r)}) = k\mu(A)$, which is our first assertion. Finally, $A^{(r)}$ is a homomorphic image of A so that $\mu^*(A^{(r)}) \leqq \mu^*(A)$, whence $\mu(A) = \mu(A^{(r)}) = \mu^*(A^{(r)}) \leqq \mu^*(A)$. As the opposite inequality $\mu^*(A) \leqq \mu(A)$ is obvious, this completes the proof of the lemma.

PROOF OF THEOREM 8. Let U be the universal ring of A. Then $A \equiv U$ and by Theorem 3 $A_k \equiv U_k$, so that it is enough to prove the theorem for U. If N is the nilradical of U then U/N has no nonzero nilpotent ideals and, since U satisfies an identity, N is locally nilpotent ([4]). Since $(U/N)_k \cong U_k/N_k$ has also no nonzero nilpotet ideals, we have by Lemma 10

$$\mu^*(U_k/N_k) = \mu^*((U/N)_k) = \mu((U/N)_k) = k\mu(U/N) = k\mu^*(U/N).$$

The proof will thus be completed if we show that $\mu^*(U_k) = \mu^*(U_k/N_k)$ and $\mu^*(U) = \mu^*(U/N)$. Let $\mathscr{F} = \Omega[x_1, x_2, \cdots]$ be the free ring over Ω in the x_i, and let $U = \mathscr{F}/T = \Omega[\bar{x}_1, \bar{x}_2, \cdots]$, where T is some T-ideal in $\mathscr{F}$ and $\bar{x}_i = x_i + T$. Suppose U/N satisfies s_n^h, so that in particular $s_n^h(\bar{x}_1, \cdots, \bar{x}_n) \in N$. Since N is nil there is a t such that $S_n^t(\bar{x}_1, \cdots, \bar{x}_n) = 0$. If $u_1, \cdots, u_n$ are arbitrary elements of U there is a homomorphism of U into U which maps $\bar{x}_i \to u_i$ and so $s_n^t(u_1, \quad, u_n) = 0$. This shows that s_n^t is an identity for U and so $\mu^*(U) \leqq \mu^*(U/N)$. Since the opposite inequality is obvious, the equality follows. The equality $\mu^*(U_k) = \mu^*(U_k/N_k)$ is proved similarly using the fact that since N is locally nilpotent, N_k is locally nilpotent and in particular nil. Here we use instead of $\bar{x}_1, \cdots, \bar{x}_n$ n generic matrices $\bar{X}^{(v)} = (\bar{x}_{ij}^{(v)})$ whose elements are taken from $\{\bar{x}_1, \bar{x}_2, \cdots\}$. The proof of Theorem 8 is now completed.

COROLLARY 11. *Let A_k satisfy a power of s_{2n} with n minimal. Then $k \mid n$ and A satisfies a power of $s_{2n/k}$.*

This is the converse of the theorem of Procesi and Small [5].

COROLLARY 12. *If A_k is an M_n-ring then $k \mid n$.*

Indeed, $\mu^*(A_k) = \mu(A_k) = n$ and the conclusion now follows from Corollary 11.

COROLLARY 13. *Assume that Ω is a field. If A is an M_n-algebra and B any algebra then $A \otimes B \equiv B_n$ and $\mu^*(A \otimes B) = \mu^*(A) \cdot \mu^*(B)$.*

The first statement follows from Theorem 1, and from this one has by Theorem 8 $\mu^*(A \otimes B) = \mu^*(B_n) = n\mu^*(B) = \mu^*(A) \cdot \mu^*(B)$.

The following was proved in [2] under different conditions.

COROLLARY 14. *Any ring satisfying an idntity with 1 as one of its coefficients, satisfies a power of a standard identity.*

PROOF. It is enough to prove the result for a universal ring U. If N is the nilradical of U, then U/N satisfies a standard identity and, as was shown in the proof of Theorem 8, U satisfies a power of it.

We close this section with an example, showing that in general there is no connection between μ and μ^*, other than the trivial relation $\mu^*(A) \leqq \mu(A)$. Let Ω be a field of characteristic 0, V an infinite dimensional vector space over Ω with basis $\{e_1, e_2, e_3, \cdots\}$, and let A be the Grassman algebra based on V. Then A has a basis consisting of 1 and the products $e_{i_1} e_{i_2} \cdots e_{i_r}$ with $i_1 < i_2 < \cdots < i_r$, and it is well known ([4], p. 260) that A satisfies the identity $[[xy]z]$ but no standard identity. Now let $A^{(n)}$ be the subalgebra of A generated by $\{1, e_1, \cdots, e_{2n-2}\}$. We assert that $\mu(A^{(n)}) = n$ while $\mu^*(A^{(n)}) = 1$. Indeed, each product $e_{i_1} \cdots e_{i_K}$ in which two of the e's are equal must vanish, whence it easily follows that $A^{(n)}$ satisfies s_{2n}. Since on the other hand we have $s_{2n-1})1, e_1, \cdots, e_{2n-2}) = s_{2n-2}(e_1, \cdots e_{2n-2}) = (2n-2)! e_1 \cdots e_{2n-2} \neq 0$, the first assertion follows. In order to prove the second assertion we turn again to the second ring of $A_{(n)}$ and its nilradical N. Each prime homomorphic image of U satisfies the identity of degree 3 $[[xy]z]$ and, being an M-ring, it therefore satisfies s_2 (i.e. it is commutative). Thus U/N satisfies s_2 and so U satisfies a power of it, that is, $\mu^*(U) = \mu^*(A) = 1$. Note that while μ^* is kept fixed, μ can be made arbitrarily large or ∞.

Reducting Identities. In this section we consider the following "downwards" problem: Assuming that A_k satisfies a given set of identities, what can we say about the identities of A? Theorem 8 solves this problem for the case of powers of standard identities, but beyond this point the above methods do not seem to apply. For instance, if A_k is an M_n-ring we do not know whether A is an M-ring. (This would imply that A is an $M_{n/k}$-ring.) So we turn to more down-to-earth methods, which naturally involve some amount of calculations. We assume that A_k satisfies an identity and show two methods of reducing this identity (in two different senses) in going down to A. e_{ij} will denote the matrix unit with 1 in its (i,j)-th place and 0 elsewhere. In what follows A will always be a ring with a unit element.

The first reduction theorem is the following

THEOREM 15. *If A_k satisfies an identity of degree n then A_{k-1} satisfies an identity of degree $n-2$.*

REMARK. The theorem still holds if instead of assuming unit element we only assume that A contains a regular element.

PROOF. By the well known linearization process we may assumee that A_k satisfies a homogeneous multilinear identity $f(x_1,\cdots,x_n)$. Write $f(x_1,\cdots,x_n) = g(x_1,\cdots,x_{n-2})x_{n-1}x_n + h(x_1,\cdots,x_n)$ where $g(x_1,\cdots,x_{n-2})x_{n-1}x_n$ is the sum of all the terms in f which end with $x_{n-1}x_n$. By suitably renaming the x_i's if necessary, we may suppose that $g \neq 0$. Note for later reference that if we happened to start with $f = s_n(x_1,\cdots,x_n)$ we would get in this process $g = s_{n-2}(x_1,\cdots,x_{n-2})$.

We now intend to show that $g(x_1,\cdots,x_{n-2})$ is an identity for A_{k-1}. Since g is multilinear it will suffice to show that $g(a_1E_1,\cdots,a_{n-2}E_{n-2}) = 0$, for every substitution of $a_i \in A$ and E_i matrix units of order $(k-1)\times(k-1)$. The E_i will also be considered as $k \times k$-matrices, by adding k-th row and column of zeroes, but note that we can then still write $g(a_1E_1,\cdots,a_{n-2}E_{n-2}) = \sum_{i,j=1}^{k-1} b_{ij}e_{ij}$ with $b_{ij}\in A$.

Consider now the following substitution from A_k in the identity $f(x_1,\cdots,x_n)$: $Q \equiv f(a_1E_1,\cdots,a_{n-2}E_{n-2},e_{lk},e_{kk})$, where $1 \leqq l < k$ is arbitrary. Since in each monomial of $h(x_1\cdots x_n)$ one of $x_1,\cdots,x_{n-2}$ appears to the right of one of x_{n-1}, x_n it is clear that $h(x_1\cdots x_n)$ vlanishes under the above substitution, and we have:

$$0 = Q = g(a_1E_1,\cdots,a_{n-2}E_{n-2})e_{lk}e_{kk} = \left(\sum_{i,j=1}^{k-1} b_{ij}e_{ij}\right)e_{lk} = \sum_{i=1}^{k-1} b_{il}e_{ik}.$$

From this we see that $b_{il} = 0$ for $i = 1,\cdots,k-1$ and since l was arbitrary, also for $l = 1,\cdots,k-1$. Thus $g(a_1E_1,\cdots,a_{n-2}E_{n-2}) = \sum b_{ij}e_{ij} = 0$, which completes the proof of the theorem.

By a remark we made during the proof of Theorem 15 we have

COROLLARY 16. *If A_k satisfies s_{2n} then A_{k-1} satisfies $s_{2(n-1)}$ and so A satisfies $s_{2(n-k+1)}$.*

COROLLARY 17. *If A_k satisfies a standard identity then $\mu(A) \leqq \mu(A_k) - k + 1$.*

A special case of Corollary 16 yields a converse to the Amitsur-Levitzki theorem [1]:

COROLARRV 18. *If A_n satisfies s_{2n} then A is commutative.*

In what follows we shall sometimes write for short $s_n(x_1,\cdots,x_n) = [x_1,\cdots,x_n]$.

LEMMA 19. *Let* $P = [a_1e_{ii}, \cdots, a_ne_{ii}, te_{ii+1}, b_1E_1, \cdots, b_rE_r]$, *where the* E_j *are matrix units whose both indices are greater than* i, *and* a_j, t, $b_j \in A$. *Then* $P = [a_1, \cdots, a_n]te_{ii+1}[b_1E_1, \cdots, b_rE_r]$.

PROOF. Let $Q = [a_1, \cdots, a_n]te_{ii+1}[b_1E_1, \cdots, b_rE_r]$. The only nonzero terms of P are among those in which the a_je_{ii} appear first in some order, then te_{ii+1} and finally the b_jE_j in some order. Conversely, each such an arrangement gives rise to some term of P. Thus, in computing P, it is sufficient to consider those permutations which are a product of a permutation of the a_je_{ii} and a permutation of the b_jE_j. The terms of P thus obtained are clearly in one-to-one correspondence with the terms of Q, and corresponding terms are equal and appear with the sign in their respective sums. Hence the equality of P and Q is established.

We conclude with a second reduction theorem.

THEOREM 20. *If* A_k *satisfies* s_m *then* A *satisfies* s_n^k *where* $n = \left[\frac{m}{k}\right]$.

PROOF. Write $m = nk + r$ with $0 \leqq r \leqq k - 1$. Since A_k satisfies s_m it certainly satisfies s_{nk+k-1}, and so we have for all a_{ij}, $t_i \in A$,

$$P \equiv [a_{11}e_{11}, \cdots, a_{1n}e_{11}, t_1e_{12}, a_{21}e_{22}, \cdots, a_{2n}e_{22}, t_2e_{23}, \cdots, t_{k-1}e_{k-1k}, a_{k1}e_{kk}, \cdots, a_{kn}e_{kk}] = 0.$$

On the other hand, applying Lemma 19 several times in succession we get:

$$\begin{aligned} P &= [a_{11}, \cdots, a_{1n}]t_1e_{12}[a_{21}, \cdots, a_{2n}]t_2e_{23}\cdots t_{k-1}e_{k-1k}[a_{k1}, \cdots, e_{kn}] \\ &= [a_{11}, \cdots, a_{1n}]t_1[a_{21}, \cdots, a_{2n}]t_2 \cdots t_{k-1}[a_{k1}, \cdots, a_{kn}]e_{1k}, \end{aligned}$$

since elements of A commute with the e_{ij}. Thus A satisfies the identity

$$[x_{11}, \cdots, x_{1n}]y_1[x_{21}, \cdots, x_{2n}]y_2 \cdots y_{k-1}[x_{k1}, \cdots, x_{kn}],$$

which upon specializing $y_1 = \cdots = y_{k-1} = 1$ and $x_{1j} = \cdots = x_{kj} = x_j$ gives the identity $[x_1, \cdots, x_n]^k$.

Note that if A_k satisfies s_m then $2\mu^*(A_k) \leqq m$ and by Theorem 8 we have $2\mu^*(A) \leqq [m/k] = n$. Hence A satisfies a power of s_r with $r \leqq n$, whence it can be easily shown that A satisfies a power of s_n. Thus the novelty in Theorem 20 is only in its specifying this power.

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